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Generalized observables in polarization optics

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Abstract
The operatorial theory of optical polarization devices can provide some of the clearest examples in which von Neumann's model of orthogonal measurement is violated. This paper analyses the global and spectral properties of some two- and three-layer optical polarizers of non-orthogonal kind by calculating the dyadic expressions of their operators in a Dirac-algebraic form. These constitute some of the most expressive examples of non-Hermitian operators corresponding to generalized observables and it is expected that the theory of generalized quantum measurement will take advantage of the simplicity of their physical realization.

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1. Introduction

One of the postulates of the standard theory of measurement in quantum mechanics is that an observable is represented by a Hermitian operator and that a measurement corresponds to a perpendicular projection of the state vector of the physical system on one of the eigenvectors of the measuring device [1–3]. This is the so-called von Neumann's model of orthogonal measurement and constitutes the basis of the standard formalism of quantum mechanics.

It is an important recent insight that in many quantum mechanical experiments that have actually been performed, the axioms of the standard formalism are violated. Among others, many measurements are not orthogonal projections [1, 3]. Correspondingly, the notion of generalized measurement was introduced in quantum mechanics [3]. A generalized formalism of quantum mechanics [3, 4] is undergoing elaboration, which includes the standard formalism as a particular case.

In the generalized formalism, the class of operators corresponding to observables is enlarged and the non-normal operators play an important role [5].

Well known and one of the clearest ways of introducing the fundamental concepts and the postulates of quantum mechanics is by analysing the interaction of the photons with the

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polarizing optical devices in (mental) photon-by-photon experiments. Dirac himself opened this way [6] and some textbooks of quantum mechanics very important from an epistemological viewpoint [7, 8], follow it, too.

So far, the optical polarizers taken into account in introducing the axioms of quantum mechanics via considerations on polarization experiments [6–9], were only of orthogonal kind.

For example, a standard linear polarizer of azimuth zero is described by the orthogonal projector:

$$\mathscr{P}_{|P_x\rangle} = |P_x\rangle\langle P_x| \tag{1}$$

its eigenvectors being $|P_x\rangle$ and $|P_y\rangle$, the x and y linear polarized states.

A standard left-handed circular polarizer is also a device of orthogonal kind. The corresponding operator:

$$\mathscr{P}_{|L\rangle} = |L\rangle\langle L| \tag{2}$$

is normal. Its eigenvectors are orthogonal: $|L\rangle$ and $|R\rangle$, the left- and right-circularly polarized states.

All the operators describing the basic ('canonical') polarizers are Hermitian, so that they describe standard observables. Moreover, the operators of all the basic devices (homogeneous polarizers, linear retarders and rotators) [10, 11] are normal operators, their eigenvectors are orthogonal, i.e. all the canonical polarization devices are of orthogonal kind.

But this is not the general situation, nor representative, in what concerns polarization devices.

First, many polarization devices are inhomogeneous (multilayer). Each layer of such a composite device is usually of orthogonal kind (the corresponding eigenstates are orthogonal). Sometimes such a multilayer polarization device is equivalent to a basic polarizer or retarder. In this case its polarization state eigenvectors are orthogonal, the corresponding operator is normal and its eigenform contains only perpendicular projectors. But often the eigenstates of a multilayer device are not orthogonal [12].

This situation is more general in polarization optics. The two eigenstates corresponding to an arbitrary direction of propagation in some crystal are generally not orthogonal [13].

In this paper I shall calculate the expressions of some simple two- and three-layer optical polarizers of non-orthogonal kind. These constitute some of the most expressive examples of non-Hermitian operators corresponding to generalized observables. The way of analysis I have chosen here is based on the spectral theory of linear operators [14, 15], expressed in a Dirac-algebraic form [16, 17]. A polar analysis will be given in a forthcoming paper.

The paper is organized as follows. In section 2 we briefly review some fundamental results of the spectral analysis of linear operators in finite-dimensional vector space. Next we calculate, on this basis, the proper and improper expressions of the operators of some basic polarization devices by means of which the non-Hermitian multilayer polarizers taken into consideration in section 3 are formed. In the last section we shall perform a detailed analysis of the external (global) and internal (spectral) properties of these non-Hermitian polarizers.

2. Proper and improper expressions of the operators of some basic polarization devices

2.1. Expansion of a linear operator in its proper (eigen-) basis and in an improper basis

Let us consider a linear operator, $\mathscr{L}$, defined on a finite-dimensional vector space and let $\{|S_i\rangle\}$, $(i = 1, \ldots, N)$ be an orthonormal basis in this space. Denoting by $\mathscr{T}$ the identity operator and making use of the closure (completeness) relation [18]:

$$\sum |S_i\rangle\langle S_i| = \mathscr{T} \tag{3}$$

we may write:

$$\mathscr{L} = \mathscr{T}\mathscr{L}\mathscr{T} = \sum_{i,j} |S_i\rangle\langle S_i|\mathscr{L}|S_j\rangle\langle S_j| = \sum_{i,j} \langle S_i|\mathscr{L}|S_j\rangle|S_i\rangle\langle S_j|. \tag{4}$$

Thus, any linear operator may be uniquely expanded in a double series of '*basic operators*' [18], $|S_i\rangle\langle S_j|$, associated with the basis $\{|S_i\rangle\}$ of the vector space on which the operator is defined.

An operator of the kind $|S_i\rangle\langle S_i|$ is a *perpendicular* projector: it gives the perpendicular projection [15] of any ket $|S\rangle$ onto the ket $|S_i\rangle$.

An operator of the kind $|S_i\rangle\langle S_j|$ $(i \neq j)$ converts the fraction $\langle S_j|S\rangle$ corresponding to the ket $|Sj\rangle$ of any ket $|S\rangle$, into a ket $|S_i\rangle$ and can be denominated *converter* (operator of conversion) or *cross-projector*. Such a converter, or cross-projector, is of orthogonal kind, the unit vectors $|S_i\rangle$ and $|S_j\rangle$ being orthogonal. The term *cross-projector* is very expressive and adequate in characterizing the action of such an operator but handled without caution it could give rise to some confusion. Obviously, a cross-projector is not idempotent, hence the cross-projectors do not pertain to the class of operators known under the name of projectors in the theory of linear operators. Particularly, in the polarization state space, the perpendicular cross projectors are nilpotents of index II.

If $|S_i\rangle$ constitutes an orthonormal basis of eigenvectors of the operator $\mathscr{L}$ itself (case in which $\mathscr{L}$ must be a normal operator):

$$\mathscr{L}|S_i\rangle = \lambda_i|S_i\rangle \tag{5}$$

$$\langle S_i|S_j\rangle = \delta_{ij} \tag{6}$$

expansion (4) becomes

$$\mathscr{L} = \sum_i \lambda_i|S_i\rangle\langle S_i| \tag{7}$$

This is the well-known expansion of a normal operator in terms of its eigenvectors and eigenvalues [18], its so-called *spectral expansion*[1] [14].

In polarization optics, the state space being two dimensional, for the device operators $\mathscr{D}$ corresponding to the polarization devices, expressions (4) and (7) reduce to

$$\mathscr{D} = \sum_{i=1}^{2}\sum_{j=1}^{2} \langle S_i|\mathscr{D}|S_j\rangle|S_i\rangle\langle S_j| \tag{8}$$

and

$$\mathscr{D} = \lambda_1|S_1\rangle\langle S_1| + \lambda_2|S_2\rangle\langle S_2|. \tag{9}$$

By using the last expression, we shall establish the expansions of some basic birefringent device (instrument) operators in their eigenrepresentations—the so-called eigen- or proper expansions of the device operators. Improper expansions of the basic device operators will be obtained too, which correspond to equation (8).

2.2. *Eigenforms of the operators of some basic polarization devices*

The basic (or 'canonical', [11]) polarization optical devices are the homogeneous polarizers and retarders. The most convenient classification of these devices, for our purposes, is that based on their eigenpolarization and eigenvalues.

[1] Here the term spectral has the sense used in the theory of linear operators.

The basic polarization device operators are normal: their eigenvectors are mutually orthogonal.

The eigenstates of an ideal elliptic homogeneous retarder are orthonormal and their associated eigenvalues are pure phase factors:

$$\lambda_1 = \exp(\mathrm{i}\delta/2) \qquad \lambda_2 = \exp(-\mathrm{i}\delta/2) \tag{10}$$

where δ is the relative retardance introduced between the fast and slow eigenstates [10].

If we label by $|E_M\rangle$ and $|E_m\rangle$ the major and the minor eigenvectors of some elliptic homogeneous retarder, by using (9) its operator may be written as

$$\mathscr{R}_{|E_M\rangle}(\delta) = |E_M\rangle\langle E_M|\exp(\mathrm{i}\delta/2) + |E_m\rangle\langle E_m|\exp(-\mathrm{i}\delta/2). \tag{11}$$

Since the eigenvalues of any elliptic ideal homogeneous retarder operator are unimodular, (10), all these operators are unitary operators.

Thus, the eigenform of the device operators of a linear retarder with azimuth of the fast axis θ is

$$\mathscr{R}_{|P_\theta\rangle}(\delta) = |P_\theta\rangle\langle P_\theta|\exp(\mathrm{i}\delta/2) + |P_{\theta+90^\circ}\rangle\langle P_{\theta+90^\circ}|\exp(-\mathrm{i}\delta/2) \tag{12}$$

For a linear retarder of azimuth zero we get

$$\mathscr{R}_{|P_x\rangle}(\delta) = |P_x\rangle\langle P_x|\exp(\mathrm{i}\delta/2) + |P_y\rangle\langle P_y|\exp(-\mathrm{i}\delta/2) \tag{13}$$

and so on.

An elliptic homogeneous ideal polarizer is characterized by orthogonal eigenstates whose associated eigenvalues are unity and zero [10]. If we denote by $|E\rangle$ the transmitted eigenstate (the principal, major, eigenvector), by virtue of (9) the device operators of an ideal elliptic polarizer may be written as

$$\mathscr{P}_{|E\rangle} = |E\rangle\langle E| \tag{14}$$

For example:

$$\mathscr{P}_{|P_x\rangle} = |P_x\rangle\langle P_x| \tag{15}$$

for a linear polarizer of azimuth zero, and

$$\mathscr{P}_{|P_\theta\rangle} = |P_\theta\rangle\langle P_\theta| \tag{16}$$

for a linear polarizer of azimuth θ.

From a mathematical viewpoint, the operators (14)–(16) are perpendicular projectors.

An elliptic homogeneous partial polarizer can be characterized by two orthogonal eigenstates $|E_l\rangle$ and $|E_h\rangle$ whose associated eigenvalues are $\lambda_l = \exp(\alpha/2)$ and $\lambda_h = \exp(-\alpha/2)$ respectively. The indices l and h refer to the low and high absorption eigenstates and α represents the relative attenuation of these eigenstates by the partial polarizer [10].

By using equation (9), the operator of an elliptic partial polarizer can be put in the form:

$$Q_{|E_l\rangle}(\alpha) = |E_l\rangle\langle E_l|\exp(\alpha/2) + |E_h\rangle\langle E_h|\exp(-\alpha/2). \tag{17}$$

This is a Hermitian operator.

All these operators, the operators of the basic ('canonical') polarization devices are normal operators. Their eigenvectors are orthogonal. Their constitutive projectors are of perpendicular kind. The *devices* are of *orthogonal* kind.

2.3. Improper expressions of the operators of some canonical polarization devices in Cartesian basis

All the expressions of the type given so far for the device operators are eigenforms of these operators (each operator is expressed in terms of its eigenprojectors and eigenvalues). The operators of linear retarders and polarizers are expanded in the corresponding rectangular bases, the operators of circular retarders and polarizers in the circular basis and those of elliptical devices in their own elliptical basis.

In order to build up a physically expressive form of a composite polarization device operator, we have to develop the calculus coherently in a unique and adequate basis, i.e. we have to transpose the expressions of all the operators of the individual constituent devices into the same, generally improper, basis.

A possibility is to start with the eigenforms of each operator—expressions of kind (9)—and to transpose each of the involved eigenvectors into the unique chosen basis, i.e. to perform a change of basis at the level of state vectors.

Another possibility is to introduce the eigenexpression of the operator in (8), i.e. to perform the change of basis directly at the level of the operators.

No matter which way we adopt, we have to make use of the relationship between the unit vectors of different orthogonal bases implied in the calculus. This relationship may be expressed either in the form of the linear relations between these unit vectors, or in the form of their scalar products.

The connection between the circular basis unit vectors $\{|R\rangle, |L\rangle\}$ and those of the Cartesian $\{|P_x\rangle, |P_y\rangle\}$ basis may be expressed in the form:

$$|R\rangle = \frac{1}{\sqrt{2}}[|P_x\rangle + \mathrm{i}|P_y\rangle] \tag{18}$$

$$|L\rangle = \frac{1}{\sqrt{2}}[|P_x\rangle - \mathrm{i}|P_y\rangle] \tag{19}$$

From (18) and (19) one also obtains

$$|P_x\rangle = \frac{1}{\sqrt{2}}[|R\rangle + |L\rangle] \tag{20}$$

$$|P_y\rangle = \frac{1}{\sqrt{2}}\mathrm{i}[|L\rangle - |R\rangle] \tag{21}$$

Generally, a linearly polarized state $|P_\theta\rangle$ may be developed in the $\{|P_x\rangle, |P_y\rangle\}$ basis as follows:

$$|P_\theta\rangle = \cos\theta|P_x\rangle + \sin\theta|P_y\rangle. \tag{22}$$

The only scalar products of the basic unit vectors we need in the following, apart from the orthogonality relations of the pair of unit vectors of the same basis, are

$$\langle P_x \mid P_\theta\rangle = \langle P_\theta \mid P_x\rangle = \cos\theta \tag{23}$$

$$\langle P_y \mid P_\theta\rangle = \langle P_\theta \mid P_y\rangle = \sin\theta. \tag{24}$$

$$\langle L \mid P_{45^\circ}\rangle = \langle P_{45^\circ} \mid L\rangle^* = \frac{1}{2}(1 + \mathrm{i}) = \frac{1}{\sqrt{2}}\,\mathrm{e}^{\mathrm{i}\pi/4} \tag{25}$$

$$\langle L \mid P_{-45^\circ}\rangle = \langle P_{-45^\circ} \mid L\rangle^* = \frac{1}{2}(1 - \mathrm{i}) = \frac{1}{\sqrt{2}}\,\mathrm{e}^{-\mathrm{i}\pi/4}. \tag{26}$$

which can be easily deduced from (18)–(22).

Starting with the eigenform of any operator, and making use of the relationship between the unit vectors of various bases, or introducing it in (8) and using the values of the scalar products of the various states, one may obtain the improper form of the operator in the chosen basis.

For example, by using (16) and (22) it is straightforward that:

$$\begin{aligned}\mathscr{P}_{|P_\theta\rangle} &= [\cos\theta|P_x\rangle + \sin\theta|P_y\rangle][\cos\theta\langle P_x| + \sin\theta\langle P_y|]\\ &= \cos^2\theta|P_x\rangle\langle P_x| + \sin\theta\cos\theta[|P_x\rangle\langle P_y| + |P_y\rangle\langle P_x|] + \sin^2\theta|P_y\rangle\langle P_y|.\end{aligned}$$

Alternatively, by using (16) in (8), the expansion of $\mathscr{P}_{|P_\theta\rangle}$ in the $\{|P_x\rangle, |P_y\rangle\}$ basis may be written as

$$\begin{aligned}\mathscr{P}_{|P_\theta\rangle} = {}&\langle P_x|P_\theta\rangle\langle P_\theta|P_x\rangle|P_x\rangle\langle P_x| + \langle P_x|P_\theta\rangle\langle P_\theta|P_y\rangle|P_x\rangle\langle P_y|\\ &+ \langle P_y|P_\theta\rangle\langle P_\theta|P_x\rangle|P_y\rangle\langle P_x| + \langle P_y|P_\theta\rangle\langle P_\theta|P_y\rangle|P_y\rangle\langle P_y|\end{aligned}$$

and, with (23) and (24):

$$\mathscr{P}_{|P_\theta\rangle} = \cos^2\theta|P_x\rangle\langle P_x| + \sin\theta\cos\theta[|P_x\rangle\langle P_y| + |P_y\rangle\langle P_x|] + \sin^2\theta|P_y\rangle\langle P_y|. \tag{27}$$

The projector $\mathscr{P}_{|P_\theta\rangle}$ is, obviously, a Hermitian operator. In this expansion, (27), the projectors $|P_x\rangle\langle P_x|$, $|P_y\rangle\langle P_y|$ are Hermitian. The cross-projectors $|P_x\rangle\langle P_y|$ and $|P_y\rangle\langle P_x|$ are not Hermitian. But they are Hermitian conjugates of each other, so that their sum is a Hermitian operator.

Similarly starting with eigenform (12) of the linear retarder of fast axis azimuth θ and making use of (22), one obtains:

$$\begin{aligned}\mathscr{R}_{|P_\theta\rangle}(\delta) &= [\cos^2\theta\exp(\mathrm{i}\delta/2) + \sin^2\theta\exp(-\mathrm{i}\delta/2)]|P_x\rangle\langle P_x|\\ &\qquad + 2\mathrm{i}\sin\theta\cos\theta\sin\frac{\delta}{2}[|P_x\rangle\langle P_y| + |P_y\rangle\langle P_x|]\\ &= \left(\cos\frac{\delta}{2} + \mathrm{i}\cos 2\theta\sin\frac{\delta}{2}\right)|P_x\rangle\langle P_x| + \mathrm{i}\sin 2\theta\sin\frac{\delta}{2}[|P_x\rangle\langle P_y|\\ &\qquad + |P_y\rangle\langle P_x|] + \left(\cos\frac{\delta}{2} - \mathrm{i}\cos 2\theta\sin\frac{\delta}{2}\right)|P_y\rangle\langle P_y|.\end{aligned} \tag{28}$$

In these improper forms it is also evident that all these operators are normal. All their constitutive projectors and converters are of perpendicular kind.

3. Non-Hermitian polarizers

3.1. Horizontal linear polarizer followed by a linear polarizer at an angle θ

This sequence of standard (orthogonal) ideal linear polarizers provides the simplest example of a non-orthogonal device in polarization optics. It is an inhomogeneous linear polarizer of azimuth θ, whose operator is, as we shall see, non-normal, and non-Hermitian.

Let us expand the operator, $\mathscr{P}_1$, of this sandwich. With (27) and (15), one obtains:

$$\begin{aligned}\mathscr{P}_1 &= \mathscr{P}_{|P_\theta\rangle}\mathscr{P}_{|P_x\rangle}\\ &= [\cos^2\theta|P_x\rangle\langle P_x| + \sin\theta\cos\theta|P_x\rangle\langle P_y| + \sin\theta\cos\theta|P_y\rangle\langle P_x| + \sin^2\theta|P_y\rangle\langle P_y|]|P_x\rangle\langle P_x|\\ &= [\cos^2\theta|P_x\rangle\langle P_x| + \sin\theta\cos\theta|P_y\rangle\langle P_x|\\ &= \cos\theta[\cos\theta|P_x\rangle + \sin\theta|P_y\rangle]\langle P_x| = \cos\theta|P_\theta\rangle\langle P_x|.\end{aligned} \tag{29}$$

Particularly for this device a simpler calculus may be done. We do not need to change the basis for one of the constituent operators:

$$\mathscr{P}_1 = \mathscr{P}_{|P_\theta\rangle}\mathscr{P}_{|P_x\rangle} = |P_\theta\rangle\langle P_\theta|P_x\rangle\langle P_x| = \cos\theta|P_\theta\rangle\langle P_x|$$

The device is a converter, which changes any incident light state, $|E\rangle$, into $|P_\theta\rangle$, through $|P_x\rangle$: the polarizer $\mathscr{P}_{|P_x\rangle}$ converts a percentage $|\langle P_x|E\rangle|^2$ of the incident light photons into its transmitted eigenstate, $|P_x\rangle$, and the polarizer $\mathscr{P}_{|P_\theta\rangle}$ converts the percentage $\cos^2\theta$ of them into the final state $|P_\theta\rangle$.

Unlike converters $|S_i\rangle\langle S_j|$ encountered in (4), the converter $|P_\theta\rangle\langle P_x|$ which occurs in (29) is not of orthogonal kind. It converts the fraction of the state $|E\rangle$ corresponding to the state $|P_x\rangle$ into the state $|P_\theta\rangle$, and these two last states are not orthogonal. The operator $|P_\theta\rangle\langle P_x|$ is a *skew converter*.

The only state which passes unchanged (apart from a factor) through this device is $|P_\theta\rangle$:

$$\mathscr{P}_1|P_\theta\rangle = \cos\theta|P_\theta\rangle\langle P_x \mid P_\theta\rangle = \cos^2\theta|P_\theta\rangle \tag{30}$$

The state $|P_y\rangle$ is blocked by the device:

$$\mathscr{P}_1|P_y\rangle = \cos\theta|P_\theta\rangle\langle P_x \mid P_y\rangle = 0 \tag{31}$$

Thus, the eigenvectors and the eigenvalues of the polarizer $\mathscr{P}_1$ are

$$|P_\theta\rangle \qquad \text{with } \lambda_1 = \cos^2\theta \tag{32}$$

$$|P_y\rangle \qquad \text{with } \lambda_2 = 0 \tag{33}$$

The two eigenvectors of the polarizer $\mathscr{P}_1$ are not orthogonal; their inner product does not vanish. The modulus of this scalar product was denoted by Lu and Chipman [12] '*inhomogeneity parameter*' and its modulus squared '*similarity factor*' by Pancharatnam [13]. For our purposes the term '*degree of non-orthogonality*' is adequate. For $\mathscr{P}_1$,

$$\eta = |\langle P_\theta \mid P_y\rangle| = \sin\theta. \tag{34}$$

In conclusion, the polarizer $\mathscr{P}_1$ is a non-orthogonal device. Its eigenvectors are (excepting the trivial case $\theta = 0$) non-orthogonal. Its operator is non-normal, and non-Hermitian. Nevertheless this polarizer prepares, for any input, the state $|P_\theta\rangle$; it corresponds to an observable: linear polarized state of the photon, of azimuth θ. In fact it gives *a non-orthogonal projection* of any input state $|E\rangle$ on its major eigenstate $|P_\theta\rangle$.

As the orthogonal polarizer (1) converts the incident photons into its transmitted eigenstate $|P_x\rangle$ or blocks them (state $|P_y\rangle$), the non-orthogonal linear polarizer (29) converts them into its transmitted eigenstate $|P_\theta\rangle$ or blocks them (state $|P_y\rangle$). Both linear polarizers project any incident state of optical polarization (SOP) in their transmitted eigenstates, $|P_x\rangle$, respectively $|P_\theta\rangle$. The only difference is that (1) performs an orthogonal projection, while (29) a non-orthogonal (oblique) one. The homogeneous linear polarizer (1) corresponds to a *standard observable*, and the inhomogeneous linear polarizer (29) corresponds to a *generalized* one.

As we shall see, a common feature of the non-orthogonal polarization devices is that their *forward- and backward-looking properties differ*.

Reversing the direction of propagation of the incoming light in this sandwich of two linear polarizers, the device operator becomes

$$\begin{aligned}
\mathscr{P}_1' &= \mathscr{P}_{|P_x\rangle}\mathscr{P}_{|P_{-\theta}\rangle} \\
&= |P_x\rangle\langle P_x|[\cos^2\theta|P_x\rangle\langle P_x| - \sin\theta\cos\theta|P_x\rangle\langle P_y| - \sin\theta\cos\theta|P_y\rangle\langle P_x| + \sin^2\theta|P_y\rangle\langle P_y|] \\
&= \cos^2\theta|P_x\rangle\langle P_x| - \sin\theta\cos\theta|P_x\rangle\langle P_y| \\
&= \cos\theta|P_x\rangle[\cos\theta\langle P_x| - \sin\theta\langle P_y|] = \cos\theta|P_x\rangle\langle P_{-\theta}|.
\end{aligned} \tag{35}$$

Thus, viewed from the opposite side, the device looks like a non-orthogonal linear polarizer of azimuth zero, which converts $\cos^2\theta|\langle P_{-\theta}|E\rangle|^2$ photons of the incident light into the state $|P_x\rangle$.

Its eigenstates and eigenvalues are, obviously:

$$|P_x\rangle \qquad \text{with } \lambda_1 = \cos\theta\langle P_{-\theta} \mid P_x\rangle = \cos^2\theta \tag{36}$$

$$|P_{90^\circ-\theta}\rangle \qquad \text{with } \lambda_2 = 0. \tag{37}$$

The degree of non-orthogonality of the two polarizers $\mathscr{P}_1$ and $\mathscr{P}_1'$ is the same:

$$\eta = |\langle P_\theta \mid P_y\rangle| = |\langle P_x|P_{90^\circ-\theta}\rangle| = \sin\theta \tag{38}$$

The linear polarizer (29)–(35) is the simplest and the most expressive example of non-orthogonal polarization devices, but it is somewhat artificial. Let us consider now some widespread polarizers of non-orthogonal kind.

3.2. Horizontal linear polarizer followed by a half-wave linear retarder at an angle $\theta/2$

Such an arrangement is used in the half-shade analyser of the polarimeters. Keeping in mind that a half-wave plate shifts the plane of polarization of the linear polarized incident light symmetrically with respect to its principal axis, it is clear that such a sandwich acts as a linear polarizer of azimuth θ. It is an inhomogeneous non-orthogonal θ-linear polarizer.

By using (28) and (15), the operator of this device may be expanded in the $\{|P_x\rangle, |P_y\rangle\}$ basis as follows:

$$\begin{aligned}\mathscr{P}_2 &= \mathscr{R}_{|P_{\theta/2}\rangle}(\pi)\mathscr{P}_{|P_x\rangle}\\ &= [\mathrm{i}\cos\theta|P_x\rangle\langle P_x| + \mathrm{i}\sin\theta|P_x\rangle\langle P_y| + \mathrm{i}\sin\theta|P_y\rangle\langle P_x| - \mathrm{i}\cos\theta|P_y\rangle\langle P_y|]|P_x\rangle\langle P_x|\\ &= \mathrm{i}\cos\theta|P_x\rangle\langle P_x| + \mathrm{i}\sin\theta|P_y\rangle\langle P_x|\end{aligned} \tag{39}$$

an expression which does not coincide with that of a homogeneous (and orthogonal) polarizer of azimuth θ (16). That is quite normal, because the sandwich described by equation (39) is not an orthogonal polarization device.

We can readily put the expansion (39) in a physically expressive form. With (16), the last form of (39) becomes:

$$\mathscr{P}_2 = \mathscr{R}_{|P_{\theta/2}\rangle}(\pi)\mathscr{P}_{|P_x\rangle} = \mathrm{i}[\cos\theta|P_x\rangle + \sin\theta|P_y\rangle]\langle P_x| \doteq |P_\theta\rangle\langle P_x| \tag{40}$$

where $\doteq$ stands for 'it is the same as'; the phase factor i has no relevance in defining the type of device.

In (40) it is obvious that the device is a skew converter: it selects the percentage $|\langle P_x|E\rangle|^2$of the incident, $|E\rangle$, photons and converts them into $|P_\theta\rangle$ photons.

The eigenvectors and the eigenvalues of the device are

$$|P_\theta\rangle \qquad \text{with } \lambda_1 = \langle P_x|P_\theta\rangle = \cos\theta \tag{41}$$

$$|P_y\rangle \qquad \text{with } \lambda_2 = 0. \tag{42}$$

Obviously, the eigenvectors are not orthogonal. This sandwich is a non-orthogonal θ linear polarizer.

If the light is propagating in a reverse direction, the device is viewed as

$$\begin{aligned}\mathscr{P}_2' &= \mathscr{P}_{|P_x\rangle}\mathscr{R}_{|P_{-\theta/2}\rangle}(\pi)\\ &= |P_x\rangle\langle P_x|[\mathrm{i}\cos\theta|P_x\rangle\langle P_x| - \mathrm{i}\sin\theta|P_x\rangle\langle P_y| - \mathrm{i}\sin\theta|P_y\rangle\langle P_x| + \mathrm{i}\cos\theta|P_y\rangle\langle P_y|]\\ &= \mathrm{i}\cos\theta|P_x\rangle\langle P_x| - \mathrm{i}\sin\theta|P_x\rangle\langle P_y|\end{aligned} \tag{43}$$

the operator and its action on the incident light differ from (39). With (22) we get:

$$\mathscr{P}_2' = \mathrm{i}|P_x\rangle\lfloor\cos\theta\langle P_x| - \sin\theta\langle P_y|\rfloor \doteq |P_x\rangle\langle P_{-\theta}| \tag{44}$$

The device is an inhomogeneous non-orthogonal linear x-polarizer, which converts $|\langle P_{-\theta} \mid E\rangle|^2$ photons of the incident light $|E\rangle$ in $|P_x\rangle$. It is a skew converter.

The eigenvectors and the eigenvalues of the device are

$$|P_x\rangle \qquad \text{with } \lambda_1 = \langle P_\theta \mid P_x\rangle = \cos\theta \tag{45}$$

$$|P_{90^\circ-\theta}\rangle \qquad \text{with } \lambda_2 = 0. \tag{46}$$

Again, the backward-looking properties of the sandwich differ from its forward-looking properties.

Both the operators $\mathscr{P}_2$ and $\mathscr{P}_2'$ are non-normal, non-Hermitian and nevertheless they correspond to observables: linear polarized states of azimuth θ, (40), and zero, (44), respectively. Their degree of non-orthogonality is the same:

$$\eta = |\langle P_\theta|P_y\rangle| = |\langle P_x \mid P_{90^\circ-\theta}\rangle| = \sin\theta. \tag{47}$$

This sandwich has a remarkable peculiarity. It allows us to define a *physical device corresponding to an orthogonal converter*. We have seen that such a converter has the general expression $|S_i\rangle\langle S_j|$, where the states $|S_i\rangle$, $|S_j\rangle$ are orthogonal. For example, $|P_y\rangle\langle P_x|$ is an orthogonal $(x \to y)$ converter.

At first sight, we could believe that a $(x \to y)$ converter could be built by crossing a linear $\mathscr{P}_{|P_x\rangle}$and a linear $\mathscr{P}_{|P_y\rangle}$ polarizer. But evidently such a device extinguishes any incident light. It is a light annihilator:

$$\mathscr{P}_{|P_y\rangle}\mathscr{P}_{|P_x\rangle} = |P_y\rangle\langle P_y \mid P_x\rangle\langle P_x| = 0$$

Let us turn back to the sandwich (40) (similar considerations are valuable for the reversed variant (44)):

For $\theta/2 = 0$ it becomes a $\mathscr{P}_{|P_x\rangle}$ polarizer (its operator becomes the orthogonal projector $|P_x\rangle\langle P_x|$).

For $\theta/2 = 45^\circ$, the device becomes a $x \to y$ converter: its operator is $|P_y\rangle\langle P_x|$. This result may be confirmed by some physical consideration. It is well known that a half-wave plate changes the azimuth of the linearly polarized incident light symmetrically with respect to its own principal axis. If the incident light is linearly polarized at 45° to this principal axis, the azimuth is changed by 90°. Thus, for $\theta/2 = 45^\circ$, the sandwich (40) acts as an orthogonal $x \to y$ converter.

It is worth pointing out that this is not the case for the previous non-orthogonal polarizer (**A**): for $\theta = 0$, $\mathscr{P}_1$ (29) becomes an orthogonal projector $|P_x\rangle\langle P_x|$, but for $\theta = 90^\circ$ it becomes a light annihilator. Mathematically this happens because of the factor $\cos\theta$ that (29) contains (and (40) does not contain). Physically this is the $x \to y$ annihilator discussed above (two crossed homogeneous linear polarizers).

3.3. Non-orthogonal two-layer circular polarizer

Circular polarizers are commonly made by laminating together a linear polarizer and a linear $\pi/2$ retarder, with the transmission direction of the polarizer at 45° to the proper ('fast' and 'slow') axes of the retarder.

If the fast axis of the retarder is the x-axis and the azimuth of the polarizer is +45° with respect to this axis, the operator of the device is

$$\mathscr{C} = \mathscr{R}_{|P_x\rangle}(\pi/2)\mathscr{P}_{|P_{45^\circ}\rangle} \tag{48}$$

With (28) for $\theta = 0$, $\delta = \pi/2$ and (27) for $\theta = 45^\circ$, this expression may be expanded, in the $\{|P_x\rangle, |P_y\rangle\}$ basis, as follows:

$$\begin{aligned}
\mathscr{C} &= \left[\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}|P_x\rangle\langle P_x| + \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}|P_y\rangle\langle P_y|\right]\frac{1}{2}[1 + |P_x\rangle\langle P_y| + |P_y\rangle\langle P_x|] \\
&= \frac{1}{2}\left[\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}|P_x\rangle\langle P_x| + \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}|P_y\rangle\langle P_y| + \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}|P_x\rangle\langle P_y| + \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}|P_y\rangle\langle P_x|\right] \\
&= \frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}|P_x\rangle[\langle P_x| + \langle P_y|] + \frac{1}{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}|P_y\rangle[\langle P_x| + \langle P_y|] \\
&= \frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\left[|P_x\rangle + \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}|P_y\rangle\right][\langle P_x| + \langle P_y|] \\
&= \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\frac{1}{\sqrt{2}}[|P_x\rangle - \mathrm{i}|P_y\rangle]\frac{1}{\sqrt{2}}[\langle P_x| + \langle P_y|]
\end{aligned}$$

and, with (19) and (22), we obtain

$$\mathscr{C}_\ell = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}|L\rangle\langle P_{45^\circ}|. \tag{49}$$

I have marked by the index ℓ the fact that the circular polarizer (49) is a left-handed one. By acting on a general elliptical SOP, $|E\rangle$:

$$\mathscr{C}_\ell|E\rangle \doteq |L\rangle\langle P_{45^\circ}|E\rangle$$

this polarizer converts the percentage $|\langle P_{45^\circ}|E\rangle|^2$ of the incoming $|E\rangle$ photons corresponding to the $|P_{45^\circ}\rangle$ state, into the final $|L\rangle$ state.

Comparing (49) with (2), it is obvious that the inhomogeneous left-circular polarizer $\mathscr{C}_\ell$ is quite another polarization device than the homogeneous left-circular polarizer $\mathscr{P}_{|L\rangle}$. From a mathematical viewpoint $\mathscr{P}_{|L\rangle}$, (2), is a perpendicular projector, whereas $\mathscr{C}_\ell$, (49), is a skew converter; the states $|P_{45^\circ}\rangle$ and $|L\rangle$ are not orthogonal.

Let us now make the spectral analysis of this converter: which are its eigenvectors and eigenvalues?

It is obvious that its major eigenvector is $|L\rangle$: the $|L\rangle$ state passes through the inhomogeneous circular polarizer (49) unchanged apart from an attenuation and phase factor. On the other hand, the state $|P_{-45^\circ}\rangle$, orthogonal to $|P_{45^\circ}\rangle$, is completely blocked by the device (by the polarizer $\mathscr{P}_{|P_{45^\circ}\rangle}$); thus the minor eigenvector of the device is $|P_{-45^\circ}\rangle$. Also making use of (25), the spectral characteristics of the device may be summarized as

$$\text{major eigenvector } |L\rangle \qquad \text{with the eigenvalue } \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\langle P_{45^\circ}|L\rangle = 1/\sqrt{2} \tag{50}$$

$$\text{minor eigenvector } |P_{-45^\circ}\rangle \qquad \text{with the eigenvalue zero.} \tag{51}$$

The two eigenvectors are not orthogonal; consequently the operator (49) is a non-normal operator. The device is of non-orthogonal kind.

For comparison, the canonical left-circular polarizer $\mathscr{P}_{|L\rangle}$, (2), is a device of orthogonal kind: its eigenvectors are $|L\rangle$ and $|R\rangle$, obviously orthogonal.

If the input face of the device is that of the polarizer, the device operator becomes

$$\mathscr{P}_3 = \mathscr{P}_{|P_{-45^\circ}\rangle}\mathscr{R}_{|P_x\rangle}(\pi/2) \tag{52}$$

With (27) for $\theta = -45^\circ$ and (28) for $\theta = 0$, $\delta = \pi/2$, we obtain further:

$$\begin{aligned}
&\frac{1}{2}[1 - |P_x\rangle\langle P_y| - |P_y\rangle\langle P_x|]\left[\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}|P_x\rangle\langle P_x| + \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}|P_y\rangle\langle P_y|\right] \\
&\qquad = \frac{1}{2}\left[\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}|P_x\rangle\langle P_x| + \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}|P_y\rangle\langle P_y| - \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}|P_x\rangle\langle P_y| - \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}|P_y\rangle\langle P_x|\right]
\end{aligned}$$

$$= \frac{1}{2}e^{i\frac{\pi}{4}}[|P_x\rangle - |P_y\rangle]\langle P_x| + \frac{1}{2}e^{-i\frac{\pi}{4}}[|P_y\rangle - |P_x\rangle]\langle P_y|$$

$$= e^{i\frac{\pi}{4}}\frac{1}{2}[|P_x\rangle - |P_y\rangle]\big[\langle P_x| - e^{-i\frac{\pi}{2}}\langle P_y|\big]$$

$$= e^{i\frac{\pi}{4}}\frac{1}{\sqrt{2}}[|P_x\rangle - |P_y\rangle]\frac{1}{\sqrt{2}}[\langle P_x| + i\langle P_y|]$$

and, with (22) for $\theta = -45^\circ$ and (19)

$$\mathscr{P}_3 = \mathscr{P}_{|P_{-45^\circ}\rangle}\mathscr{R}_{|P_x\rangle}(\pi/2) = e^{i\frac{\pi}{4}}|P_{-45^\circ}\rangle\langle L|. \tag{53}$$

This time, viewed from the opposite side, the device looks like a non-orthogonal linear polarizer.

Its eigenvectors and eigenvalues are

$$|P_{-45^\circ}\rangle \qquad \text{with } \lambda_1 = e^{i\frac{\pi}{4}}\langle L|P_{-45^\circ}\rangle = 1/\sqrt{2} \tag{54}$$

$$|R\rangle \qquad \text{with } \lambda_2 = 0 \tag{55}$$

where we have made use of (26).

In conclusion, the widespread inhomogeneous (two-layer) circular polarizer is another example of a non-orthogonal polarization device.

Its forward-looking features differ from the backward-looking ones. In one direction it acts really as a circular polarizer $\mathscr{C}_\ell$, (49), but in the opposite direction it acts as a linear polarizer $\mathscr{P}_3$, (53).

From a spectral viewpoint (inner properties of the operators), in both variants the eigenvectors of this sandwich are non-orthogonal, with the same degree of non-orthogonality:

$$\eta = |\langle L|P_{-45^\circ}\rangle| = |\langle P_{-45^\circ}|R\rangle| = \frac{1}{\sqrt{2}} \tag{56}$$

From the viewpoint of its global (external) properties, in both cases the device reduces (apart from a complex factor) to a skew converter.

From its spectral properties as well as from its global properties it is also evident that viewed from both sides the device is of non-orthogonal kind, and its operators are non-normal:

$$\mathscr{C}_\ell\mathscr{C}_\ell^\dagger \neq \mathscr{C}_\ell^\dagger\mathscr{C}_\ell \qquad \mathscr{P}_3\mathscr{P}_3^\dagger \neq \mathscr{P}_3^\dagger\mathscr{P}_3 \tag{57}$$

non-Hermitian:

$$\mathscr{C}_\ell \neq \mathscr{C}_\ell^\dagger \qquad \mathscr{P}_3 \neq \mathscr{P}_3^\dagger \tag{58}$$

where the dagger superscript stands for adjoint.

But each of these operators corresponds to an observable (left-circular polarized state and linear polarized -45° state, respectively).

Finally we have to note that a more general, elliptic, non-orthogonal polarizer is obtained if, instead of a $\mathscr{P}_{|P_{45^\circ}\rangle}$, a $\mathscr{P}_{|P_\theta\rangle}$ linear polarizer is used in (48) [19, 20]. The non-orthogonality of this polarizer was expressed by Berry and Klein ([19], appendix A) by pointing out that the extinguished state of the device is not orthogonal to the transmitted one, but to the state passed by the linear polarizer, a statement to which our results (50) and (51) correspond.

Almost all the operators of the polarizers we have analysed above are *simple* (they have a complete set of eigenvectors, here two) and *singular* (one of the eigenvalues is zero). This is the case for all the homogeneous ideal orthogonal polarizers (1), (2), (14)–(16), as well as for the non-orthogonal polarizers (29), (35), (40), (44), (49), (53). All these operators have two distinct eigenvectors and, for each of them, one of the eigenvectors is annihilated by the corresponding operator.

But we have encountered, as a particular case of one of the previously discussed devices (end of section 3.2), a very special device: the orthogonal converter

$$|P_y\rangle\langle P_x|. \tag{59}$$

This device has only one eigenvector, $|P_y\rangle$, which is annihilated by its operator. The operator (59) is no longer a simple one, it is *defective* (*degenerate*) and, at the same time, *singular*.

We have imagined the device corresponding to the operator (59)—more or less artificially—as a special case of the non-orthogonal polarizition device (40), for illustrating the physical realization of an orthogonal converter. In the following we shall analyse the operator of a widespread polarization device which pertains to this very special class of defective singular operators.

3.4. Ambidextrous three-layer circular polarizer

One can make an ambidextrous circular polarizer by sandwiching a linear polarizer at 45° between two suitable oriented 90° linear retarders [20].

Let us consider the following arrangement:

$$\mathscr{R}_{|P_x\rangle}(\pi/2)\mathscr{P}_{|P_{45^\circ}\rangle}\mathscr{R}_{|P_x\rangle}(\pi/2). \tag{60}$$

With (12) for $\theta = 0$ and (27) for $\theta = 45^\circ$, the operator (60) may be developed in the $\{|P_x\rangle, |P_y\rangle\}$ basis as follows:

$$\begin{aligned}
&\left[e^{i\frac{\pi}{4}}|P_x\rangle\langle P_x| + e^{-i\frac{\pi}{4}}|P_y\rangle\langle P_y|\right]\frac{1}{2}[1 + |P_x\rangle\langle P_y| + |P_y\rangle\langle P_x|]\left[e^{i\frac{\pi}{4}}|P_x\rangle\langle P_x| + e^{-i\frac{\pi}{4}}|P_y\rangle\langle P_y|\right]\\
&\quad= \frac{1}{2}\left[e^{i\frac{\pi}{4}}|P_x\rangle\langle P_x| + e^{-i\frac{\pi}{4}}|P_y\rangle\langle P_y|\right]\left[e^{i\frac{\pi}{4}}|P_x\rangle\langle P_x| + e^{-i\frac{\pi}{4}}|P_y\rangle\langle P_y|\right.\\
&\qquad\left. + e^{-i\frac{\pi}{4}}|P_x\rangle\langle P_y| + e^{i\frac{\pi}{4}}|P_y\rangle\langle P_x|\right]\\
&\quad= \frac{1}{2}\left[|P_x\rangle\langle P_y| + e^{i\frac{\pi}{2}}|P_x\rangle\langle P_x| + |P_y\rangle\langle P_x| + e^{-i\frac{\pi}{2}}|P_y\rangle\langle P_y|\right]\\
&\quad= \frac{i}{2}|P_x\rangle[\langle P_x| - i\langle P_y|] + \frac{1}{2}|P_y\rangle[\langle P_x| - i\langle P_y|]\\
&\quad= i\frac{1}{\sqrt{2}}[|P_x\rangle - i|P_y\rangle]\frac{1}{\sqrt{2}}[\langle P_x| - i\langle P_y|]
\end{aligned}$$

$$\mathscr{R}_{|P_x\rangle}(\pi/2)\mathscr{P}_{|P_{45^\circ}\rangle}\mathscr{R}_{|P_x\rangle}(\pi/2) = i|L\rangle\langle R|. \tag{61}$$

Thus, the three-layer device (60) and (61) is a $|R\rangle \to |L\rangle$ converter and gives, for any input, left-circularly polarized output. Any incoming SOP is converted by the device into left-circular light. It is evident from (61) that this device has only one eigenstate, namely $|L\rangle$, and for this state the eigenvalue is zero, i.e. the unique eigenstate of the device is completely blocked. Polarization devices having only one eigenpolarization are said to have *degenerate anisotropy* [10]. The device (61) is a *degenerate polarizer*.

If the light comes from the opposite side, the same three-layer sandwich looks like

$$\mathscr{R}_{|P_x\rangle}(\pi/2)\mathscr{P}_{|P_{-45^\circ}\rangle}\mathscr{R}_{|P_x\rangle}(\pi/2) \tag{62}$$

which, again with (12) and (27), may be written successively as

$$\begin{aligned}
&\left[e^{i\frac{\pi}{4}}|P_x\rangle\langle P_x| + e^{-i\frac{\pi}{4}}|P_y\rangle\langle P_y|\right]\frac{1}{2}[1 - |P_x\rangle\langle P_y| - |P_y\rangle\langle P_x|]\left[e^{i\frac{\pi}{4}}|P_x\rangle\langle P_x| + e^{-i\frac{\pi}{4}}|P_y\rangle\langle P_y|\right]\\
&\quad= \frac{1}{2}\left[e^{i\frac{\pi}{4}}|P_x\rangle\langle P_x| + e^{-i\frac{\pi}{4}}|P_y\rangle\langle P_y|\right]\left[e^{i\frac{\pi}{4}}|P_x\rangle\langle P_x| + e^{-i\frac{\pi}{4}}|P_y\rangle\langle P_y|\right.
\end{aligned}$$

$$- \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}|P_x\rangle\langle P_y| - \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}|P_y\rangle\langle P_x|\big]$$

$$= \frac{1}{2}\big[-|P_x\rangle\langle P_y| + \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}|P_x\rangle\langle P_x| - |P_y\rangle\langle P_x| + \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}|P_y\rangle\langle P_y|\big]$$

$$= \frac{\mathrm{i}}{2}[|P_x\rangle - \mathrm{i}|P_y\rangle]\langle P_x| - \frac{1}{2}[|P_x\rangle - \mathrm{i}|P_y\rangle]\langle P_y|$$

$$= \mathrm{i}\frac{1}{\sqrt{2}}[|P_x\rangle + \mathrm{i}|P_y\rangle]\frac{1}{\sqrt{2}}[\langle P_x| + \mathrm{i}\langle P_y|] \equiv \mathrm{i}|R\rangle\langle L|.$$

In conclusion:

$$\mathscr{R}_{|P_x\rangle}(\pi/2)\mathscr{P}_{|P_{-45^\circ}\rangle}\mathscr{R}_{|P_x\rangle}(\pi/2) = \mathrm{i}|R\rangle\langle L|. \tag{63}$$

This is a $|L\rangle \to |R\rangle$ converter and gives, for any input, right-circularly polarized light. Its eigenstate is unique, $|R\rangle$, and the corresponding eigenvalue is zero.

Thus, the backward-looking properties of the device differ from the forward-looking ones. From one side it is a degenerate left-circular polarizer, and from the other side it is a degenerate right-circular polarizer; globally—an *ambidextrous degenerate circular polarizer*.

When, instead of $\mathscr{P}_{|P_{45^\circ}\rangle}$, one uses in (60), more generally, a $\mathscr{P}_{|P_\theta\rangle}$ linear polarizer, an *ambidextrous degenerate* elliptical polarizer is obtained.

4. Conclusions. Non–Hermitian polarizers in quantum measurements

In the standard formalism of quantum mechanics a measurement corresponds to a Hermitian operator. The eigenvalues of this operator are the measurement results and the probabilities are determined by the expectation values of the orthogonal projections on its eigenvectors. These are the postulates of the Dirac–von Neumann formalism of quantum measurement.

A pertinent illustration of this formalism is given by the SOP measurement of the photons by means of various standard (orthogonal) polarizers. Such a measurement corresponds to an orthogonal projection of the initial SOP on the major eigenvector of the measuring device (the polarizer).

It is by now well known that in many quantum mechanical experiments von Neumann's model of orthogonal measurement is violated, the measurement does not correspond to an orthogonal projection; it does not have a standard character.

Once again the optical polarization can provide some very expressive examples of devices and (SOP) measurement experiments which do not have an orthogonal but a generalized character.

It turns out that the eigenvectors of some well-known inhomogeneous polarizers are not orthogonal, i.e. these polarizers are not orthogonal devices, their operators are not Hermitian. Consequently, a SOP measurement with such a device, preparative (using the device as polarizer) as well as determinative (when the device plays the role of an analyser) does not correspond to an orthogonal projection, is not a standard measurement but a generalized one.

In this paper I have presented and analysed in a quantum mechanical language the global and spectral properties of some of these non-Hermitian polarizers. The analysis is given on the basis of the spectral theory of linear operators, which is most directly connected with the problem of quantum measurement. A polar analysis—which was suggested by the manuscript referee and will constitute the object of a later paper—could give a further insight into the properties of these non-Hermitian polarizers and, more general, of non-orthogonal polarization devices.

The non-Hermitian character of these polarizers is emphasized by the analysis of internal (spectral) as well as external (global) properties of their operators. The eigenvectors are not orthogonal. The operators are not idempotent, i.e. are not orthogonal projectors.

The reversed devices are also polarizers, but their global and spectral properties differ strongly from those of the direct devices. In contrast, for a canonical polarizer these properties are identical; such a polarizer is reversible.

What we have to point out, as an essential conclusion, is that in polarization optics there are plenty of devices of non-orthogonal kind. Moreover, this is the natural situation in this field: the two eigenvectors corresponding to an arbitrary direction of propagation in a crystal are usually non-orthogonal. It is expected that the generalized formalism of quantum mechanics will take advantage of this fact in its future development.

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